Factor Maps as Patterns on U^2

Hutter RNN Project

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1 The Universal Model

Recall:

$$U = (E, T, P, f, \omega)$$

- E = event space
- $T \subseteq \mathbb{N}^{|E|}$ = thought vectors (log-space)
- P = patterns (transition structure)
- $f : T \times E \to T$ = integration
- ω = prior over patterns

2 Factor Maps

2.1 Definition

A **factor map** is a morphism $\pi : U \to U'$ where $U' = (E', T', P', f', \omega')$ is a coarser model.

Concretely, π consists of:

1. $\pi_E : E \to E'$ (event projection)
2. $\pi_T : T \to T'$ (thought projection)
3. Compatibility: $\pi_T(f(t, e)) = f'(\pi_T(t), \pi_E(e))$

2.2 Example: ES Projection

$$\pi_{ES} : \text{Bytes} \to \text{ES} = \{\text{Digit, Punct, Vowel, White, Other}\}$$

$$\pi_{ES}(\text{'a'}) = \text{Vowel} \quad (1)$$
$$\pi_{ES}(\text{' '}) = \text{White} \quad (2)$$
$$\pi_{ES}(\text{'7'}) = \text{Digit} \quad (3)$$

This induces a factor map on the full model.

3 U^2: The Product Model

3.1 Construction

Given $U = (E, T, P, f, \omega)$, define:

$$U^2 = U \times U = (E \times E, T \times T, P^2, f^2, \omega^2)$$

where:

- $E \times E$ = pairs of events (joint event space)
- $T \times T$ = pairs of thoughts
- P^2 = patterns on pairs
- $f^2((t_1, t_2), (e_1, e_2)) = (f(t_1, e_1), f(t_2, e_2))$

3.2 Patterns on U^2

A **pattern on** U^2 is an element of P^2, representing a relationship between two events.

In log-space:

$$p_{(e_1,e_2)\to(e'_1,e'_2)} \in \mathbb{N}$$

This is the log-support for the joint transition.

4 Factor Maps as Patterns

4.1 Key Insight

A factor map $\pi : U \to U'$ **is** a pattern on U^2.

```
U  ---->  U'
|          |
f          f'
|          |
v          v
U  ---->  U'
```

The diagram commutes: $\pi \circ f = f' \circ \pi$.

The factor map π relates:

- $(e, \pi(e)) \in E \times E'$ — event and its image
- $(t, \pi(t)) \in T \times T'$ — thought and its image

4.2 Pattern Representation

Encode the factor map as a pattern matrix:

$$\Pi_{e,e'} = \begin{cases} 1 & \text{if } \pi(e) = e' \\ 0 & \text{otherwise} \end{cases}$$

In log-space ($\mathbb{N}$):

$$\Pi_{e,e'} = \begin{cases} \infty & \text{if } \pi(e) = e' \\ 0 & \text{otherwise} \end{cases}$$

(Infinite support for valid mappings, zero for invalid.)

4.3 Soft Factor Maps

For probabilistic/learned factor maps:

$$\Pi_{e,e'} = \log P(e'|e)$$

This is exactly the embedding construction from earlier!

5 Composition

5.1 Factor Map Composition

Given $\pi_1 : U \to U'$ and $\pi_2 : U' \to U''$:

$$\pi_2 \circ \pi_1 : U \to U''$$

In pattern notation:

$$(\Pi_2 \circ \Pi_1)_{e,e''} = \max_{e'} \min(\Pi_1[e, e'], \Pi_2[e', e''])$$

This is tropical matrix multiplication (max-min in log-space).

5.2 The Factor Lattice

Factor maps form a lattice:

```
            U
         /  |  \
    _ES  _byte  _pos
      /     |      \
   U_ES   U_byte   U_pos
      \     |      /
        \   |   /
         U_top
```

where:

- U_{ES} = 5-event ES model
- U_{byte} = identity (256 events)
- U_{pos} = positional model
- $U_{\top}$ = trivial 1-event model

6 Learning Factor Maps

6.1 From Data

Given observations, learn the soft factor map:

$$\hat{\Pi}_{e,e'} = \log \frac{\text{count}(e, e')}{\text{count}(e)}$$

This is the conditional log-probability (the embedding).

6.2 From Hidden States

For an RNN with hidden state h:

$$\Pi_{h \to e} = W_{out} \cdot h$$

The output weights define a (soft) factor map from hidden space to event space.

7 Summary

Concept	Representation
Factor map $\pi : U \to U'$	Pattern on $U \times U'$
Hard projection	Binary pattern matrix
Soft/learned projection	Log-probability matrix
Composition	Tropical matrix multiply
Embedding	Row of factor map matrix

Punchline: Factor maps, embeddings, and patterns are all the same thing viewed differently. They all live in U^2.