

# Factor Maps as Patterns on $U^2$

Hutter RNN Project

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## 1 The Universal Model

Recall:

$$U = (E, T, P, f, \omega)$$

- $E$  = event space
- $T \subseteq \mathbb{N}^{|E|}$  = thought vectors (log-space)
- $P$  = patterns (transition structure)
- $f : T \times E \rightarrow T$  = integration
- $\omega$  = prior over patterns

## 2 Factor Maps

### 2.1 Definition

A **factor map** is a morphism  $\pi : U \rightarrow U'$  where  $U' = (E', T', P', f', \omega')$  is a coarser model.

Concretely,  $\pi$  consists of:

1.  $\pi_E : E \rightarrow E'$  (event projection)
2.  $\pi_T : T \rightarrow T'$  (thought projection)
3. Compatibility:  $\pi_T(f(t, e)) = f'(\pi_T(t), \pi_E(e))$

### 2.2 Example: ES Projection

$$\pi_{ES} : \text{Bytes} \rightarrow \text{ES} = \{\text{Digit}, \text{Punct}, \text{Vowel}, \text{White}, \text{Other}\}$$

$$\pi_{ES}('a') = \text{Vowel} \tag{1}$$

$$\pi_{ES}(' ') = \text{White} \tag{2}$$

$$\pi_{ES}('7') = \text{Digit} \tag{3}$$

This induces a factor map on the full model.

### 3 $U^2$ : The Product Model

#### 3.1 Construction

Given  $U = (E, T, P, f, \omega)$ , define:

$$U^2 = U \times U = (E \times E, T \times T, P^2, f^2, \omega^2)$$

where:

- $E \times E$  = pairs of events (joint event space)
- $T \times T$  = pairs of thoughts
- $P^2$  = patterns on pairs
- $f^2((t_1, t_2), (e_1, e_2)) = (f(t_1, e_1), f(t_2, e_2))$

#### 3.2 Patterns on $U^2$

A **pattern on  $U^2$**  is an element of  $P^2$ , representing a relationship between two events.

In log-space:

$$p_{(e_1, e_2) \rightarrow (e'_1, e'_2)} \in \mathbb{N}$$

This is the log-support for the joint transition.

### 4 Factor Maps as Patterns

#### 4.1 Key Insight

A factor map  $\pi : U \rightarrow U'$  is a pattern on  $U^2$ .

$$\begin{array}{ccc} U & \dashrightarrow & U' \\ | & & | \\ f & & f' \\ | & & | \\ v & & v \\ U & \dashrightarrow & U' \end{array}$$

The diagram commutes:  $\pi \circ f = f' \circ \pi$ .

The factor map  $\pi$  relates:

- $(e, \pi(e)) \in E \times E'$  — event and its image
- $(t, \pi(t)) \in T \times T'$  — thought and its image

## 4.2 Pattern Representation

Encode the factor map as a pattern matrix:

$$\Pi_{e,e'} = \begin{cases} 1 & \text{if } \pi(e) = e' \\ 0 & \text{otherwise} \end{cases}$$

In log-space ( $\mathbb{N}$ ):

$$\Pi_{e,e'} = \begin{cases} \infty & \text{if } \pi(e) = e' \\ 0 & \text{otherwise} \end{cases}$$

(Infinite support for valid mappings, zero for invalid.)

## 4.3 Soft Factor Maps

For probabilistic/learned factor maps:

$$\Pi_{e,e'} = \log P(e'|e)$$

This is exactly the embedding construction from earlier!

## 5 Composition

### 5.1 Factor Map Composition

Given  $\pi_1 : U \rightarrow U'$  and  $\pi_2 : U' \rightarrow U''$ :

$$\pi_2 \circ \pi_1 : U \rightarrow U''$$

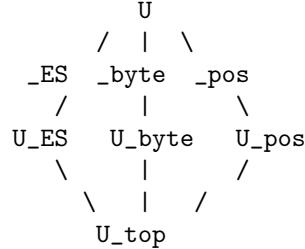
In pattern notation:

$$(\Pi_2 \circ \Pi_1)_{e,e''} = \max_{e'} \min(\Pi_1[e, e'], \Pi_2[e', e''])$$

This is tropical matrix multiplication (max-min in log-space).

### 5.2 The Factor Lattice

Factor maps form a lattice:



where:

- $U_{ES}$  = 5-event ES model
- $U_{byte}$  = identity (256 events)
- $U_{pos}$  = positional model
- $U_{\top}$  = trivial 1-event model

## 6 Learning Factor Maps

### 6.1 From Data

Given observations, learn the soft factor map:

$$\hat{\Pi}_{e,e'} = \log \frac{\text{count}(e, e')}{\text{count}(e)}$$

This is the conditional log-probability (the embedding).

### 6.2 From Hidden States

For an RNN with hidden state  $h$ :

$$\Pi_{h \rightarrow e} = W_{out} \cdot h$$

The output weights define a (soft) factor map from hidden space to event space.

## 7 Summary

Concept	Representation
Factor map $\pi : U \rightarrow U'$	Pattern on $U \times U'$
Hard projection	Binary pattern matrix
Soft/learned projection	Log-probability matrix
Composition	Tropical matrix multiply
Embedding	Row of factor map matrix

**Punchline:** Factor maps, embeddings, and patterns are all the same thing viewed differently. They all live in  $U^2$ .