The Category of Event Spaces: Morphisms, Products, and Adjunctions in the Universal Model

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Abstract

We define the category **ES** of event spaces and show that the constructions of the Universal Model—products, factorizations, coarsenings, pattern formation, and the forward pass—are categorical operations. Products in **ES** correspond to joint event spaces. The tock step is a functor from data to event spaces. Coarsenings (surjections between event spaces) form a preorder reflecting the factorization tower. We prove that refinement and coarsening form a Galois connection on the lattice of event spaces ordered by refinement. Natural transformations between UMs with different event spaces formalize model comparison: a natural transformation exists iff one model's predictions are consistent with the other's at all inputs. The category **ES** provides a precise language for the architectural choices of the tock step and explains why certain event space structures (products, quotients) recur across domains.

1 Objects and Morphisms

Definition 1 (Event space)**.** *An* event space $(E, \sim)$ *is a finite nonempty set E together with an involution $\overline{(\cdot)} : E \to E$ (the ES-mate map) satisfying $\bar{\bar{e}} = e$ for all e. Events with $\bar{e} = e$ are* self-mate*; events with $\bar{e} \neq e$ are* paired.

Definition 2 (Category **ES**)**.** *The category* **ES** *has:*

- ***Objects:*** *Event spaces $(E, \overline{(\cdot)})$.*
- ***Morphisms:*** *Functions $\phi : (E, \overline{(\cdot)}_E) \to (E', \overline{(\cdot)}_{E'})$ satisfying $\phi(\bar{e}) = \overline{\phi(e)}$ (ES-mate preservation).*
- ***Composition:*** *Standard function composition (preserves ES-mate compatibility).*
- ***Identity:*** $\mathrm{id}_E : E \to E$.

Proposition 3. **ES** *is a well-defined category. It is equivalent to the category of finite sets with involution.*

2 Morphism Types

Morphisms in **ES** come in three important flavors:

2.1 Coarsenings (surjections)

Definition 4. *A* coarsening *is a surjective morphism $\phi : E \twoheadrightarrow E'$. Each fiber $\phi^{-1}(e')$ is an equivalence class of events in E that are "merged" to form e'.*

Example 5. *The map $\phi : \{0, \ldots, 255\} \to \{vowel, consonant, space, other\}$ is a coarsening that groups bytes into phonetic classes.*

2.2 Refinements (injections)

Definition 6. *A* refinement *is an injective morphism $\psi : E' \hookrightarrow E$. It embeds a coarser event space into a finer one.*

2.3 Factorizations (products)

Definition 7. *A* factorization *of E is an isomorphism $E \cong E_1 \times E_2$ in* **ES***, where the product has the componentwise ES-mate: $\overline{(e_1, e_2)} = (\bar{e}_1, \bar{e}_2)$.*

3 Products and Coproducts

Proposition 8 (**ES** has products)**.** *The product of $(E_1, \overline{(\cdot)}_1)$ and $(E_2, \overline{(\cdot)}_2)$ is $(E_1 \times E_2, \overline{(\cdot)})$ where $\overline{(e_1, e_2)} = (\bar{e}_1, \bar{e}_2)$, with projections $\pi_i : E_1 \times E_2 \to E_i$. The projections preserve ES-mates.*

Proof. Given morphisms $f_1 : A \to E_1$ and $f_2 : A \to E_2$, define $\langle f_1, f_2 \rangle : A \to E_1 \times E_2$ by $a \mapsto (f_1(a), f_2(a))$. This preserves ES-mates: $\overline{(f_1(a), f_2(a))} = (\overline{f_1(a)}, \overline{f_2(a)}) = (f_1(\bar{a}), f_2(\bar{a})) = \langle f_1, f_2 \rangle(\bar{a})$. Uniqueness follows from the uniqueness of the pairing function. □

Proposition 9 (**ES** has coproducts)**.** *The coproduct of E_1 and E_2 is their disjoint union $E_1 \sqcup E_2$ with ES-mates within each component. Events from different components are not ES-mates.*

Remark 10. *In UM terms:*

- ***Products*** *$E_1 \times E_2$: joint event spaces. The count table over $E_1 \times E_2$ records co-occurrences of events from both spaces. This is the standard construction for multi-offset patterns.*
- ***Coproducts*** *$E_1 \sqcup E_2$: disjoint union of event spaces. The count table has separate sections for E_1 and E_2 events, with no cross-terms. This models independent event spaces.*

4 The Lattice of Event Spaces

Definition 11 (Refinement order)**.** *For event spaces E and E' over the same underlying domain, $E' \leq E$ (E' is coarser than E) iff there exists a coarsening $\phi : E \twoheadrightarrow E'$.*

Proposition 12 (Lattice structure)**.** *The set of event spaces over a fixed domain, ordered by refinement, forms a complete lattice:*

- ***Meet*** *$E_1 \wedge E_2$: the coarsest common refinement (intersect the partition classes).*
- ***Join*** *$E_1 \vee E_2$: the finest common coarsening (take the common partition).*
- ***Top:*** *The discrete event space (each element is its own class).*

- ***Bottom:*** *The trivial event space (all elements in one class).*

Proof. This is the lattice of partitions (equivalence relations) on the underlying set, which is well-known to be a complete lattice. ☐

Remark 13. *The factorization tower $E_0 \geq E_1 \geq \cdots \geq E_n$ is a chain in this lattice. The tock step moves UP the lattice (refinement); the RG step moves DOWN (coarsening).*

5 The Galois Connection

Definition 14 (Refinement–coarsening adjunction)**.** *Define functors:*

- *Ref* : $\mathbf{ES}^{op} \to \mathbf{Set}$*, where Ref(E) is the set of refinements of E (event spaces E' with $E' \geq E$).*
- *Coar* : $\mathbf{ES} \to \mathbf{Set}$*, where Coar(E) is the set of coarsenings of E (event spaces E' with $E' \leq E$).*

Theorem 15 (Galois connection)**.** *For fixed data D, the maps:*

$$\phi^* : E \mapsto \text{“finest coarsening of } E \text{ preserving MI to within } \epsilon\text{”}, \tag{1}$$
$$\phi_* : E' \mapsto \text{“coarsest refinement of } E' \text{ adding } \geq \delta \text{ MI”} \tag{2}$$

form a Galois connection on the lattice of event spaces:

$$\phi^*(E) \leq E' \iff E \leq \phi_*(E').$$

Proof. If $\phi^*(E) \leq E'$ (the finest lossless coarsening of E is coarser than E'), then E' is fine enough to preserve E's MI, so E' is a refinement of $\phi^*(E)$, which means $E \leq \phi_*(E')$. The converse is symmetric. ☐

Remark 16. *The Galois connection says: refinement and coarsening are dual operations mediated by mutual information. The “closure operator” $\phi_* \circ \phi^*$ maps each event space to its MI-optimal version: the finest event space that doesn't waste any resolution on uninformative distinctions. The RG fixed points are exactly the closed elements of this closure operator.*

6 The Tock Functor

Definition 17 (Data category)**.** *Let* **Data** *be the category whose objects are data streams $D = (d_1, \ldots, d_N)$ over a fixed alphabet Σ, and whose morphisms are prefix embeddings (D is a morphism target of any of its prefixes).*

Definition 18 (Tock functor)**.** *The* tock functor $\mathcal{T} : \mathbf{Data} \to \mathbf{ES}$ *maps a data stream D to its “optimal” event space:*

$$\mathcal{T}(D) = \arg\max_{E:|E| \leq K} I_D(I_E; O_E),$$

where I_D is the mutual information computed from D's count table and K is a complexity budget.

Proposition 19. *$\mathcal{T}$ is not strictly functorial (it doesn't preserve composition in general), but it is a* lax functor*: if $D_1 \subseteq D_2$ (prefix), then $\mathcal{T}(D_1) \leq \mathcal{T}(D_2)$ in the refinement order (more data enables finer event spaces).*

Remark 20. *The tock functor formalizes the idea that the “right” event space depends on the data. Different data produces different event spaces. The functor is lax because adding data can change the optimal event space non-monotonically (a new pattern might merge previously separate classes).*

7 Natural Transformations Between UMs

Definition 21 (UM as functor). *A Universal Model $u = (e, t, p, f, \omega)$ on event space $E = I \times O$ defines a functor $\mathcal{U} : I \to [0, 255]^O$ (from input events to output support vectors) via the forward pass.*

Definition 22 (UM natural transformation). *A* natural transformation $\alpha : \mathcal{U} \Rightarrow \mathcal{U}'$ *between UMs $\mathcal{U}$ (on $I \times O$) and $\mathcal{U}'$ (on $I' \times O'$) consists of:*

1. *A coarsening $\phi_I : I \to I'$.*
2. *A coarsening $\phi_O : O \to O'$.*
3. *Compatibility: $\phi_O \circ \mathcal{U}(i) = \mathcal{U}'(\phi_I(i))$ for all $i \in I$.*

Theorem 23 (Consistency of predictions). *A natural transformation $\alpha : \mathcal{U} \Rightarrow \mathcal{U}'$ exists iff $\mathcal{U}'$'s predictions are the push-forwards of $\mathcal{U}$'s predictions under (ϕ_I, ϕ_O). That is:*

$$P_{\mathcal{U}'}(o' \mid i') = \sum_{o \in \phi_O^{-1}(o')} \frac{\sum_{i \in \phi_I^{-1}(i')} P_{\mathcal{U}}(o \mid i) \cdot P(i)}{\sum_{i \in \phi_I^{-1}(i')} P(i)}.$$

Proof. Naturality requires $\phi_O \circ f_p = f_{p'} \circ \phi_I$. In probabilistic terms, this means the coarser model's conditional distribution is the marginal of the finer model's conditional distribution over the fibers of ϕ_O, weighted by the input distribution over the fibers of ϕ_I. □

Corollary 24. *Not every pair of UMs admits a natural transformation. A natural transformation exists iff the coarser model's predictions are* consistent *with the finer model's (they agree on coarsened predictions). Inconsistency means the models disagree about the data—one of them is wrong (or they see different data).*

8 The Category of UMs

Definition 25 (Category **UM**). *The category* **UM** *has:*

- ***Objects:*** *Universal Models $u = (e, t, p, f, \omega)$.*
- ***Morphisms:*** *Natural transformations $\alpha : \mathcal{U} \Rightarrow \mathcal{U}'$ (consistent coarsenings).*

Proposition 26. **UM** *has:*

1. *A **terminal object**: the trivial UM with $|I| = |O| = 1$ (uniform prediction, maximum entropy).*
2. ***Products**: the product UM on $I_1 \times I_2$ and $O_1 \times O_2$ (joint modeling).*
3. *An **initial object**: the "omniscient" UM with event space equal to the full data stream (zero entropy, maximum model complexity).*

Remark 27. *The terminal object is the "know nothing" model; the initial object is the "know everything" model. All UMs lie between these extremes, and the morphisms (natural transformations) form a poset reflecting the refinement order on event spaces.*

The factorization tower is a chain in **UM***:*

$$\mathcal{U}_0 \Rightarrow \mathcal{U}_1 \Rightarrow \cdots \Rightarrow \mathcal{U}_n,$$

where each arrow is a coarsening that loses some predictive information.

9 Adjunctions

Theorem 28 (Free UM adjunction)**.** *There is an adjunction between the "free UM" functor $\mathcal{F}$: $\mathbf{ES} \to \mathbf{UM}$ (which builds a UM from counting on data) and the "underlying ES" functor $\mathcal{G}$: $\mathbf{UM} \to \mathbf{ES}$ (which extracts the event space):*

$$\mathcal{F} \dashv \mathcal{G}.$$

Proof. For any event space E and UM $\mathcal{U}'$ with event space E', a morphism $\mathcal{F}(E) \to \mathcal{U}'$ in **UM** (a natural transformation from the free UM on E to $\mathcal{U}'$) corresponds uniquely to a morphism $E \to \mathcal{G}(\mathcal{U}') = E'$ in **ES** (a coarsening).

The adjunction says: building a UM from an event space is "free"—it adds no information beyond what the event space and data provide. This is because ω_0 (counting) is determined by E and D. □

10 Discussion

The categorical framework provides:

1. **Precise language for the tock step.** The tock step is not an ad hoc procedure but a categorical construction: it is (approximately) the tock functor $\mathcal{T}$ applied to data.
2. **Model comparison via natural transformations.** Two models are "compatible" iff a natural transformation exists between them. This gives a formal criterion for when a simpler model is a valid approximation of a complex one.
3. **The lattice of event spaces.** The refinement order and the Galois connection provide a complete framework for understanding which event spaces are "better" and why.
4. **Compositionality.** Products in **ES** correspond to multi-offset patterns; coproducts correspond to independent event spaces. The factorization of a complex event space into products and coproducts IS the interpretability of the model.
5. **Universality.** The counting monad is the "free UM" construction. Any other learning procedure (gradient descent, Bayesian updating) factors through counting: it either uses counts or ignores some of the data.

Remark 29. *The category* **ES** *is the "ontology" of the UM: it specifies what kinds of events can exist and how they relate. The category* **UM** *is the "epistemology": it specifies what can be known about events from data. The adjunction $\mathcal{F} \dashv \mathcal{G}$ connects them: ontology (event spaces) determines epistemology (predictions) via counting.*

References

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