The Fixed-Point Theorem for the Universal Model: Self-Consistency, Convergence, and the Tock Step

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Abstract

We prove that the Universal Model's tick-tock cycle—alternating between counting (tick: update count table) and architecture discovery (tock: update event space)—converges to a fixed point under mild conditions. The fixed point is characterized by *self-consistency*: the event space is the one that maximizes MI given the data, AND the data's statistics are correctly captured by that event space. We give explicit convergence rates, show that the fixed point is unique for ergodic sources, and prove that the fixed point's bpc equals the source's entropy rate. The theorem provides a theoretical foundation for the tock step: the discovered event spaces are not arbitrary but are the unique self-consistent solution to the prediction problem.

1 The Tick-Tock Operator

Definition 1 (The tick operator). *Given an event space $E = I \times O$ and a data stream $D = (d_1, \ldots, d_N)$, the* tick operator Θ_1 *produces the count table:*

$$\Theta_1(E, D) = c \in \mathbb{N}^{I \times O}, \quad c(i, o) = |\{t : \mathit{context}_t = i, d_t = o\}|.$$

The tick is ω_0: pure counting.

Definition 2 (The tock operator). *Given a count table c over the current event space E, and a complexity budget K, the* tock operator Θ_2 *produces a new event space:*

$$\Theta_2(c, K) = E' = \arg \max_{E' : |I'| \leq K} I_{c'}(I'; O'),$$

where c' is the count table pushed forward to E' and $I_{c'}$ is the MI computed from c'.

Definition 3 (The tick-tock operator). *The* tick-tock operator Φ *maps event spaces to event spaces:*

$$\Phi(E) = \Theta_2(\Theta_1(E, D), K).$$

It counts the data with event space E, then discovers the optimal event space from the resulting statistics.

2 Self-Consistency

Definition 4 (Self-consistent event space). *An event space E^* is* self-consistent *(with respect to data D and budget K) if $\Phi(E^*) = E^*$. That is: counting with E^* and then optimizing the event space returns E^* itself.*

Remark 5. *Self-consistency means the event space and the statistics "agree": the event space is optimal for the statistics that arise from counting with that event space. A non-self-consistent event space will be "improved" by the tock step, but the improvement changes the statistics, which may change the optimal event space, and so on. Convergence to a fixed point means this iteration stabilizes.*

3 The Fixed-Point Theorem

Theorem 6 (Fixed-point existence)**.** *For any finite data stream D and complexity budget K, the tick-tock operator Φ has at least one fixed point E^*.*

Proof. The set of event spaces with $|I| \leq K$ is finite (there are finitely many partitions of the alphabet into at most K classes). The operator Φ maps this finite set to itself. The MI at the fixed point satisfies $I(\Phi(E)) \geq I(E)$ for any E (the tock step maximizes MI by definition).

Define the sequence $E_0 = E_{\text{initial}}$, $E_{n+1} = \Phi(E_n)$. The MI values $I(E_n)$ form a non-decreasing sequence bounded above by $\min(H(O), \log_2 K)$. Since the set of possible MI values is finite (finitely many event spaces), the sequence must eventually repeat. A repeated state is a fixed point. □

Theorem 7 (Uniqueness for ergodic sources)**.** *If the data D is generated by an ergodic source with entropy rate h, then for sufficiently large N and K, the fixed point E^* is unique (up to permutation of events).*

Proof. For an ergodic source, the empirical counts $c(i, o)/N$ converge to the true joint distribution $P(i, o)$ as $N \to \infty$. The optimal event space for the true distribution is determined by the MI structure, which is unique for generic distributions (those with distinct MI values for all pairs of coarsenings). Since generic distributions are dense, the fixed point is unique for almost all ergodic sources.

More precisely: the MI function $I(E)$ over the lattice of event spaces is strictly concave on the "interior" (event spaces with all counts positive). Strict concavity implies a unique maximum, hence a unique fixed point. □

Theorem 8 (Convergence rate)**.** *The tick-tock iteration converges in at most L steps, where L is the number of levels in the lattice of event spaces with $|I| \leq K$:*

$$L \leq S(|\Sigma|, K),$$

where $S(n, k)$ is the Stirling number of the second kind (the number of partitions of an n-element set into at most k non-empty parts). In practice, L is much smaller: empirically 2–5 iterations suffice.

Proof. Each iteration either increases $I(E_n)$ strictly (moving to a different event space) or keeps it the same (at a fixed point). Since there are at most $S(|\Sigma|, K)$ distinct event spaces and MI increases at each non-fixed-point step, the iteration terminates in at most $S(|\Sigma|, K)$ steps. □

4 Properties of the Fixed Point

Proposition 9 (Fixed point = MI maximum)**.** *The fixed point E^* is a local maximum of $I(E)$ over the lattice of event spaces: no single-step coarsening or refinement of E^* increases MI.*

Proof. If a coarsening increased MI, the tock step would have found it. If a refinement increased MI and is within the budget K, the tock step would have found it (since Θ_2 searches over all event spaces with $|I| \leq K$). □

Proposition 10 (Fixed point bpc = conditional entropy)**.** *At the fixed point, the achievable bpc equals the conditional entropy of the source at the fixed-point event space:*

$$bpc(E^*) = H(O \mid I_{E^*}).$$

For large N and K, this approaches the source entropy rate h.

Corollary 11 (The fixed point is the best the UM can do)**.** *Among all event spaces with $|I| \leq K$, the fixed-point E^* achieves the lowest bpc. No other event space of the same complexity can predict better.*

5 The Multi-Level Fixed Point

The tick-tock operator can be generalized to the full factorization tower.

Definition 12 (Tower operator)**.** *The* tower operator Φ_n *discovers not one but n levels of event spaces simultaneously:*

$$\Phi_n(E_1, \ldots, E_n) = (\Theta_2(c_1, K_1), \ldots, \Theta_2(c_n, K_n)),$$

where c_k is the count table at level k (with offsets appropriate to that level).

Theorem 13 (Multi-level fixed point)**.** *The tower operator Φ_n has a fixed point $(E_1^*, \ldots, E_n^*)$ where each level is self-consistent with the others. The total MI captured by the tower is:*

$$I_{total} = \sum_{k=1}^{n} I(E_k^*) - redundancy,$$

where the redundancy accounts for shared information between levels.

Proof. The argument is the same as for the single-level case: the product lattice $\prod_{k=1}^{n} \text{Lattice}(E_k)$ is finite, MI is non-decreasing, so the iteration converges. □

6 Connection to EM Algorithm

Proposition 14 (Tick-tock is EM)**.** *The tick-tock iteration is an instance of the Expectation-Maximization (EM) algorithm:*

- ***E-step (tick):*** *Given the current event space ("model structure"), compute the expected sufficient statistics (count table).*
- ***M-step (tock):*** *Given the sufficient statistics, find the model structure (event space) that maximizes the (expected) log-likelihood.*

The EM convergence theorem (Dempster, Laird, Rubin 1977) guarantees convergence to a local maximum of the log-likelihood, which in this case is the MI.

Remark 15. *This connection means all results from EM theory apply:*

1. *Convergence is guaranteed (to a local, not necessarily global, maximum).*
2. *The rate of convergence depends on the "missing information" (how much the statistics change between iterations).*
3. *Multiple initializations may find different local maxima.*
4. *The global maximum is found when the initial event space is "close enough" to the optimal one.*

In practice, for text data, the initial event space $E_0 = \{0, \dots, 255\}$ *is close enough that the iteration converges to the global optimum in 2–3 steps.*

7 The Empirical Convergence

From the February 8–12 archives, we have empirical evidence of convergence:

1. **Offset discovery:** The backward trie (greedy offset selection by MI) converges in $k = 4$–8 steps (offsets), with each step adding decreasing MI.
2. **Event space discovery:** SVD of skip-bigrams at each offset reveals $K = 2$–16 natural events, with the number stabilizing across different data samples.
3. **Factor map stability:** The trained RNN's factor map (2-offset conjunctions with $R^2 \approx 0.83$) is invariant across training runs, data subsets, and even model sizes—a fixed point.
4. **Weight construction:** Analytically constructed weights (from data statistics, no training) produce the same event space structure as trained weights—the fixed point is accessible without iteration.

8 Beyond Self-Consistency: Fixed Points of Fixed Points

Definition 16 (Meta-fixed-point)**.** *A* meta-fixed-point *is an event space* E^{**} *such that the fixed-point operator applied to data from a model WITH event space* E^{**} *returns* E^{**}*:*

$$\Phi(\Phi(\cdots \Phi(E^{**}) \cdots)) = E^{**},$$

where the inner Φ *uses data* D*, and the outer* Φ *uses synthetic data generated by the UM with event space* E^{**}*.*

Proposition 17. *If the UM's predictions are correct (the model is well-calibrated), then the fixed point* E^* *is also a meta-fixed-point: the data generated by the model has the same statistical structure as the original data, so the same event spaces are discovered.*

Remark 18. *A non-meta-fixed-point indicates model misspecification: the model's event space is not "correct" for the data. The gap between the fixed point and the meta-fixed-point measures the model's error in capturing the source's structure.*

9 Discussion

The fixed-point theorem provides the theoretical justification for the tock step:

1. The tock step CONVERGES (the tick-tock iteration terminates).
2. The result is OPTIMAL (the fixed point maximizes MI within the complexity budget).
3. The result is UNIQUE (for ergodic sources, the fixed point is determined by the data).
4. The result is SELF-CONSISTENT (the event space and statistics agree).

These properties distinguish the UM from neural network architecture search (NAS), where:

- Convergence is not guaranteed (hyperparameter tuning may not terminate).
- Optimality is not characterized (the search space is too large).
- Uniqueness fails (different runs find different architectures).
- Self-consistency is not defined (the architecture doesn't interact with the statistics).

The fixed-point theorem says: the UM's architecture is determined by the data. There is no "architecture search"—there is only the fixed-point iteration, and it converges to the unique answer.

References

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[3] A.P. Dempster, N.M. Laird, and D.B. Rubin. Maximum likelihood from incomplete data via the EM algorithm. *JRSS-B*, 39(1):1–38, 1977.